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Summable families in tempered distribution spaces

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Abstract

In this note we define summable families in tempered distribution spaces and state some their properties and characterizations. Summable families are the analogous of summable sequences in separable Hilbert spaces, but in tempered distribution spaces having elements (functional) realizable as generalized vectors indexed by real Euclidean spaces (not pointwise defined ordered families of scalars indexed by real Euclidean spaces in the sense of distributions). Any family we introduce here is summable with respect to every tempered system of coefficients belonging to a certain normal spaces of distributions, in the sense of superpositions. The summable families we present in this note are one possible rigorous and simply manageable mathematical model for the infinite families of vector-states appearing in the formulation of the continuous version of the celebrated Principle of Superpositions in Quantum Mechanics.

1 Characterizations of ${}^{\mathcal{D}}$families (*)

As we proved the summability theorem of Schwartz families and characterization of summability, in a perfectly analogous way, it can be proved the following theorem.

The ${}^{\mathcal{D}}$families in $\mathcal{D}'_n$ can be defined analogously to the ${}^{\mathcal{S}}$families in $\mathcal{S}'_n$.

Definition (family of tempered distributions of class $\mathcal{D}$). *Let v be a family of distributions in the space $\mathcal{D}'_n$ indexed by the Euclidean space $\mathbb{R}^m$.*

The family v is called a ***Schwartz family of class*** $\mathcal{S}$ *or even* $^{\mathcal{D}}$***family*** *if, for each test function $\phi \in \mathcal{D}_n$, the image of the test function ϕ by the family v - that is the function $v(\phi) : \mathbb{R}^m \to \mathbb{K}$ defined by*

$$v(\phi)(p) := v_p(\phi),$$

for each index $p \in \mathbb{R}^m$ - belongs to the space of test functions $\mathcal{D}_m$. We shall denote the set of all $^{\mathcal{D}}$families by $\mathcal{D}(\mathbb{R}^m, \mathcal{D}'_n)$.

The following theorem holds since the Corollary of page 91 of Dieudonné-Schwartz seminal paper holds true because $\mathcal{D}'_n$ is an $\mathcal{LF}$-space.

Theorem (basic properties on $^{\mathcal{D}}$families). *Let $v \in \mathcal{D}(\mathbb{R}^m, \mathcal{D}'_n)$ be a family of distributions. Then the following assertions hold and are equivalent:*

1) *for every $a \in \mathcal{D}'_m$ the composition $u = a \circ \widehat{v}$, i.e., the functional*

$$u : \mathcal{D}_n \to \mathbb{K} : \phi \mapsto a\left(\widehat{v}(\phi)\right),$$

is a distribution;

2) *the operator $\widehat{v}$ is transposable;*

3) *the operator $\widehat{v}$ is $(\sigma(\mathcal{D}_n), \sigma(\mathcal{D}_m))$-continuous from $\mathcal{D}_n$ to $\mathcal{D}_m$;*

4) *the operator $\widehat{v}$ is a strongly continuous from $(\mathcal{D}_n)$ to $(\mathcal{D}_m)$.*

2 Algebraic ^{E}Families and ^{E}summable families

Let us begin with a family of tempered distribution which is not of class $\mathcal{S}$.

Example (a family that is not of class $\mathcal{S}$). Let u be a distribution in $\mathcal{S}'_n$ and let v be the constant family in $\mathcal{S}'_n$, indexed by the Euclidean space $\mathbb{R}^m$, defined by $v_y = u$, for each point $y \in \mathbb{R}^m$. Then, if the distribution u

is different from zero, v is not of class $\mathcal{S}$. In fact, let $\phi \in \mathcal{S}(\mathbb{R}^n, \mathbb{K})$ be such that $u(\phi) \neq 0$; for every point-index $y \in \mathbb{R}^m$, we have

$$\begin{aligned} v(\phi)(y) &= v_y(\phi) = \\ &= u(\phi)1_{\mathbb{R}^m}(y), \end{aligned}$$

where, $1_{\mathbb{R}^m}$ is the constant $\mathbb{K}$-functional on $\mathbb{R}^m$ with value 1. Thus, the function $v(\phi)$ is a constant $\mathbb{K}$-functional on $\mathbb{R}^m$ different from zero, and so it cannot live in the space $\mathcal{S}_m$.

The preceding example induces us to consider other classes of families in addition to the ${}^{\mathcal{S}}$families, for this reason, we will give the following definitions.

We shall denote by $\mathcal{C}_m$ the space $\mathcal{C}^0(\mathbb{R}^m, \mathbb{K})$ of continuous functions defined on the Euclidean space $\mathbb{R}^m$ and with values in the scalar field $\mathbb{K}$.

Definition (^{E}families and algebraically ^{E}summable families). *Let E be a subspace of the function space $\mathcal{F}(\mathbb{R}^m, \mathbb{K})$ (without any topology) containing the space $\mathcal{S}_m$. If v is a family in the distribution space $\mathcal{S}'_n$ indexed by $\mathbb{R}^m$, we say that the family v is an* ^{E}***family*** *if, for every test function ϕ in $\mathcal{S}_n$, the image $v(\phi)$ of the test function by the family v lies in the subspace E. An ^{E}family v is said to be* ***algebraically*** ^{E}***summable*** *or ${}^{E^*}$summable if, for every functional a in the algebraic dual E^*, the functional*

$$\int_{\mathbb{R}^m} av : \mathcal{S}_n \to \mathbb{K} : \phi \mapsto a(v(\phi))$$

is a tempered distribution living in $\mathcal{S}'_n$. More generally, if F is a part of the algebraic dual E^* we say that the family v is ^{F}summable if the above functional $\phi \mapsto a(v(\phi))$ is a tempered distribution living in $\mathcal{S}'_n$, for every functional a in F.

Example. With the preceding new definition, the family of the above example is a ${}^{\mathcal{E}_m}$family in $\mathcal{S}'_n$, where by $\mathcal{E}_m$ we (in standard way) denote the space $C^\infty(\mathbb{R}^m, \mathbb{K})$ of smooth function from $\mathbb{R}^m$ into $\mathbb{K}$.

Moreover, for every tempered distribution a in $\mathcal{E}'_m$, we have

$$\begin{aligned} a(v(\phi)) &= a(u(\phi)1_{\mathbb{R}^m}) = \\ &= u(\phi)a(1_{\mathbb{R}^m}) = \\ &= u(\phi)\int_{\mathbb{R}^m} a, \end{aligned}$$

where by $\int_{\mathbb{R}^m} a$ we denote the integral of the distribution a (we recall that the compact support distributions are integrable and their integrals is defined as their value on the constant unit functional $1_{\mathbb{R}^m}$), so that

$$\int_{\mathbb{R}^m} av = \left(\int_{\mathbb{R}^m} a \right) u,$$

and the family v is ${}^{\mathcal{E}'_m}$summable.

3 ^{E}Families and ^{E}summable families

Remark. Let E be a subspace of the function space $\mathcal{F}(\mathbb{R}^m, \mathbb{K})$ (without any topology) containing the space $\mathcal{S}_m$ and let w be a Hausdorff locally convex topology on the subspace E.

- If the topological vector space $(\mathcal{S}_m)$ is continuously imbedded in the space E_w, then, the topological dual E'_w is continuously imbedded in the space $\mathcal{S}'_m$. In this case, in Distribution Theory, we say that the dual E'_w is *a space of tempered distribution on* $\mathbb{R}^m$.
- Moreover, if the topological vector space E_w is continuously imbedded in the space $(\mathcal{C}_m)$, then the dual $\mathcal{C}'_m$ is contained in the dual E'_w. In other terms, every Radon measure on $\mathbb{R}^m$ with compact support belongs to the dual E'_w and, in particular, the Dirac family is contained in the topological dual E'_w. Since the Dirac basis in sequentially total in the space of tempered distributions $\mathcal{S}'_m$ (the linear hull of the Dirac basis is dense in $\mathcal{S}'_m$) and since the space E'_w is continuously imbedded in the space $\mathcal{S}'_m$ itself, the Dirac family shall be sequentially total also in the topological vector space $(E'_w)_\sigma$, that is sequentially dense with respect to the weak* topology $\sigma(E', E)$.

Now we can give two new definitions.

Definition (^{E}families and ^{E}summable families). *Let E be a subspace of the space $\mathcal{F}(\mathbb{R}^m, \mathbb{K})$ containing the space $\mathcal{S}_m$ and endowed with a locally convex linear topology w. If v is a family in the distribution space $\mathcal{S}'_n$ indexed*

*by $\mathbb{R}^m$. We say that the family v is an ^{E}**family** if, for every test function ϕ in $\mathcal{S}_n$, the image $v(\phi)$ of the test function by the family v lies in the subspace E. An ^{E}family is said to be ^{E}**summable** if for every tempered distribution a in the topological dual E'_w the functional*

$$\mathcal{S}_n \to \mathbb{K} : \phi \mapsto a(v(\phi))$$

is a tempered distribution in $\mathcal{S}'_n$.

4 Normal spaces of distributions and summability

Definition (of normal space of test function for $\mathcal{S}'_m$). *We will call a locally convex topological vector space E a **normal space of test functions for the distribution space** $\mathcal{S}'_m$ if it verifies the following properties*

- *the space E is an algebraic subspace of the space $\mathcal{C}_m$;*
- *the space E contains the space $\mathcal{S}_m$;*
- *the topological vector space $(\mathcal{S}_m)$ is continuously imbedded and dense in the topological vector space E;*
- *the topological vector space E is continuously imbedded in the space $(\mathcal{C}_m)$.*

*In these conditions the dual E' is called **a normal space of tempered distributions on** $\mathbb{R}^m$.*

Theorem (on the ^{E}family generated by a linear and continuous operator). *Let E be a normal space of test functions for the space $\mathcal{S}'_m$, let $A : \mathcal{S}_n \to E$ be a linear and continuous operator of the space $(\mathcal{S}_n)$ into the space E and let δ be the Dirac family in $\mathcal{C}'_m$. Then, the family of functionals*

$$A^\vee := (\delta_p \circ A)_{p\in\mathbb{R}^m}$$

is a family of distribution in $\mathcal{S}'_n$ and it is an ^{E}family.

We can prove that:

Theorem. *Let E be a normal space of test functions for the distribution space $\mathcal{S}'_m$. Then, every ^{E}family in $\mathcal{S}'_n$ (obviously indexed by the m-dimensional Euclidean space) is ^{E}summable.*

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